UNIT-4 : Complex Variable- Differentiation

Complex variable (Introduction): $x+iy$ is a complex variable & it is denoted by z, so $z = x+iy$, where $i = \sqrt{-1}$

Different forms of complex variable:

1) $z = x+iy$ $\longrightarrow$ (Cartesian form)

2) $z = r(\cos\theta + i\sin\theta)$ $\longrightarrow$ (Polar form)

where $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(y/x)$

3) $z = re^{i\theta}$ $\longrightarrow$ (Exponential form)

$e^{i\theta} = \cos\theta + i\sin\theta$

& $e^{-i\theta} = \cos\theta - i\sin\theta$.

Function of a complex variable: $f(z)$ is a function of a complex variable z & it is denoted by w.

so $\boxed{w = f(z) = u+iv}$

where u & v are real & imaginary parts of $f(z)$. u & v are functions of x & y.

Limit of a function of a Complex variable: let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then the limit of $f(z)$ as z approaches z_0 is w_0.

$$\boxed{\lim_{z\to z_0} f(z) = w_0}$$

Example: Prove that $\lim\limits_{z \to (1-i)} \dfrac{z^2+4z+3}{z+1} = 4-i$

Sol^n: $\lim\limits_{z \to (1-i)} \dfrac{z^2+4z+3}{z+1} = \lim\limits_{z \to (1-i)} \dfrac{(z+1)(z+3)}{(z+1)} = \lim\limits_{z \to (1-i)} (z+3)$

$= 1-i+3 = 4-i$ Ans.

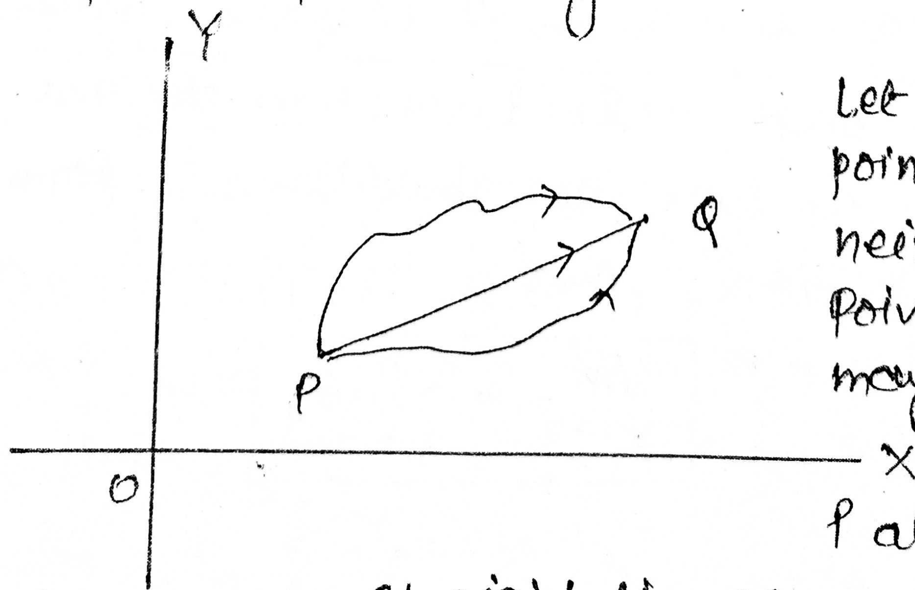
Continuity: If $f(z)$ is a function of complex variable z. Then $f(z)$ is said to be continuous at $z=z_0$ if

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$$

Differentiability: Let $f(z)$ be a single valued function of the variable z, then

$$\boxed{f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}}$$

provided that the limit exists & is independent of the path along which $\delta z \to 0$.



Let P be a fixed point & Q be a neighbouring point. Then Q may approach P along any straight line or curved path.

Question: Prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner, where $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$ when $z \neq 0$

$= 0$ when $z = 0$

(2012-13)(2018-19)

Solution: we have $\frac{f(z)-f(0)}{z} = \frac{1}{(x+iy)}\left[\frac{x^3y(y-ix)}{x^6+y^2} - 0\right]$

$= \frac{1}{(x+iy)} \cdot \frac{x^3y(-i^2y-ix)}{(x^6+y^2)} = \frac{-i\, x^3y\cancel{(x+iy)}}{\cancel{(x+iy)}\,(x^6+y^2)}$

$= \frac{-i\,x^3y}{x^6+y^2}$

$\boxed{\text{case-I}}$ Along Any Radius vector $\to$ $y = mx$ then.

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3(mx)}{x^6+m^2x^2} = \lim_{x\to 0} \frac{-imx^2}{x^4+m^2}$

$= 0$ —— (1)

$\boxed{\text{case-II}}$ Along a curve $y = x^3 \to$

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-ix^3x^3}{x^6+x^6} = -\frac{i}{2}$ —— (2)

By (1) it is clear that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector

By (2) $\frac{f(z)-f(0)}{z}$ does not tending to zero as $z \to 0$

$\Rightarrow$ $f'(0)$ does not exist, $f(z)$ is not differentiable at $z=0$

Analytic function: A function $f(z)$ is said to be analytic at a point z_0, if $f(z)$ is differentiable not only at z_0 but every point of some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.

The point at which the function is not analytic is called 'singular point' of the function.

An Analytic function is also known as 'Holomorphic', 'Regular' & Monogenic.

Entire function: A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

Note 1) An entire function is always analytic, differential & continuous. But converse is not true.

2) Analytic function is always differentiable & continuous. But converse is not true.

3) A Differentiable function is always continuous. But converse is not true.

Necessary And Sufficient conditions for $f(z)$ to be Analytic:

Necessary Conditions → Let $w = f(z) = u(x,y) + iv(x,y)$, then necessary conditions for $f(z)$ to be analytic at all the points are

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

(ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

} are known as Cauchy-Riemann equations [C-R equations] in cartesian form.

provided $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ exist.

Sufficient condition: Sufficient condition for $f(z)$ to be analytic at all the point is that

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous functions of x & y

at all the points.

Ques: Define analytic function with an example. (2017-18)

Sol^n: Analytic function: A single valued function is said be analytic at a point z_0 if it has a derivative not only at z_0 but also in some nbd of z_0.

(Exp): $f(z) = z^3$ is analytic everywhere. (2018-19)

Sol^n → $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) = u + iv$

$\Rightarrow u = x^3 - 3xy^2$ & $v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial u}{\partial y} = -6xy$, $\frac{\partial v}{\partial x} = 6xy$, $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Thus C-R equations are satisfied at every point & all partial derivatives are continuous everywhere

So $f(z) = z^3$ is analytic everywhere.

Ques : Define analytic function to discuss the analyticity of $f(z) = Re(z^3)$ in the complex plane. (2014)

Sol^n : $f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$ (As Above)

$f(z) = Re(z^3) = x^3 - 3xy^2$

here $u = x^3 - 3xy^2$ & $v = 0$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial u}{\partial y} = -6xy$, $\frac{\partial v}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$

$\Rightarrow$ C-R eq^{n}s are not satisfied so

$f(z) = Re(z^3)$ is not analytic function.

Ques : Prove that function $e^x(\cos y + i\sin y)$ is analytic & find its derivative. (2015-16)

OR

The function $f(z) = e^x(\cos y + i\sin y)$ is holomorphic or not. (2018-19)

Sol^n : let $f(z) = u + iv = e^x(\cos y + i\sin y)$

$\Rightarrow u = e^x\cos y$ & $v = e^x\sin y$

$\frac{\partial u}{\partial x} = e^x\cos y$, $\frac{\partial u}{\partial y} = -e^x\sin y$

$\frac{\partial v}{\partial x} = e^x\sin y$, $\frac{\partial v}{\partial y} = e^x\cos y$

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\Rightarrow$ C-R eq^{n}s are satisfied.

& all partial derivatives of u & v are continuous everywhere. therefore f(z) is analytic.

Now if f(z) is analytic then

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y)$

$f'(z) = e^{x} \cdot e^{iy} = e^{x+iy} = e^{z}$

$f'(z) = e^{z}$ Ans.

Ques : find the values of c_1 & c_2 such that
$f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$ is analytic.
Also find $f(z)$. (2016-17)

Sol^n : Here $f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$

$\Rightarrow u = x^2 + c_1 y^2 - 2xy$ & $v = c_2 x^2 - y^2 + 2xy$

$\frac{\partial u}{\partial x} = 2x - 2y$, $\frac{\partial v}{\partial y} = -2y + 2x$

$\frac{\partial u}{\partial y} = 2c_1 y - 2x$, $\frac{\partial v}{\partial x} = 2c_2 x + 2y$

C-R equations are , $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$2x - 2y = -(2y) + 2x$ (which is true)

&

$2c_1 y - 2x = -(2c_2 x + 2y)$

comparing coefficients of x & y

$2c_1 = -2$ & $-2 = -2c_2$

$\Rightarrow c_1 = -1$ & $c_2 = +1$

$f(z) = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$

Now $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$= (2x - 2y) + i(2x + 2y)$

$= 2(x + iy) + 2i(x + iy) = 2z + 2iz$

$f'(z) = 2(1+i)z$ Ans.

Ques : Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin. (2013-14)

Sol^n : Given that $f(z) = \log z = \log(x+iy)$

We know that $x+iy = re^{i\theta}$ where $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\frac{y}{x}$

$f(z) = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta \log e = \log r + i\theta$

$f(z) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$ — ①

so $u = \frac{1}{2}\log(x^2+y^2)$ & $v = i\tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$ & $\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$

so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

C-R equations are satisfied except at origin.

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x-iy}{x^2+y^2}$

$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{z}$

$f'(z)$ does not exist at origin.

$\Rightarrow f(z)$ is analytic everywhere except at origin.

Ques : Prove that $f(z) = \sinh z$ is analytic. (2016-17).

Sol^n: Given that $f(z) = \sinh z$

$f(z) = \sinh(x+iy)$

$= \sinh x \cos y + i \cosh x \sin y$

$\Rightarrow u = \sinh x \cos y$

$v = \cosh x \sin y$

$\frac{\partial u}{\partial x} = \cosh x \cos y, \ \frac{\partial u}{\partial y} = -\sinh x \sin y$

$\frac{\partial v}{\partial x} = \sinh x \sin y, \ \frac{\partial v}{\partial y} = \cosh x \cos y$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$

Thus C-R equations are satisfied & all partial derivatives of I^st order are continuous so $f(z) = \sinh z$ is analytic everywhere.

Some Results

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\cosh x = \frac{e^{x} + e^{-x}}{2}$

$\sinh x = \frac{e^{x} - e^{-x}}{2}$

$\sin(ix) = i \sinh x$

$\cos(ix) = \cosh x$

$D(\cosh x) = \sinh x$

$D(\sinh x) = \cosh x$

$\cosh(ix) = \cos x$

$\sinh(ix) = i \sin x$

Derivative of $\sinh z$:

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y$

$\cosh x \cosh iy + \sinh x \sinh iy$

$= \cosh(x+iy) = \cosh z$

$\boxed{\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B}$

$\sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy$

$= \sinh x \cos y + i \cosh x \sin y.$

To check Analyticity at origin ⇒

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k}$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h}$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k}$$

$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$ —①, $f'(0)$ exists if limit ① exists uniquely along any path.

Ques : Show that for the function given as

$$f(z) = \begin{cases} \frac{2xy(x+iy)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

The C-R equations are satisfied at origin but derivative of $f(z)$ does not exist at origin. (2015-16)

Soln : $f(z) = u + iv = \frac{2xy(x+iy)}{x^2+y^2}$, $z \neq 0$

& $f(z) = 0$ when $z = 0$

$$\Rightarrow u = \frac{2x^2y}{x^2+y^2} \quad \& \quad v = \frac{2xy^2}{x^2+y^2}$$

At origin $\Rightarrow$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,k)-v(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\Rightarrow$ C-R equations are satisfied at origin.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \left[\frac{\frac{2xy(x+iy)}{x^2+y^2} - 0}{(x+iy)}\right]$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy}{x^2+y^2}$$

taking $z\to 0$ along x-axis $\Rightarrow y=0$ then

$$f'(0) = \lim_{x\to 0} 0 = 0$$

now taking $z\to 0$ along a line $y=x$ then

$$f'(0) = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1$$

which shows that $f'(0)$ does not exist uniquely along any path.
C-R eq^{n}s are satisfied at origin but $f(z)$ is not analytic at origin.

Ques : Show that the function $f(z) = \frac{x^3 y^5 (x+iy)}{x^6 + y^{10}}$, $z \neq 0$ & $f(0) = 0$ is not analytic at origin even though it satisfies C-R equation at the origin. (2014-15)

Sol^n : Here $f(z) = u + iv = \frac{x^3 y^5 (x+iy)}{x^6 + y^{10}}$, $z \neq 0$

& $f(0) = 0$

So $u = \frac{x^4 y^5}{x^6 + y^{10}}$ & $v = \frac{x^3 y^6}{x^6 + y^{10}}$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^6}}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = 0$$

from the above result, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at origin.

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x^3y^5(x+iy)-0}{x^6+y^{10}}\right]\frac{1}{(x+iy)}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y^5}{x^6+y^{10}}$$

$y^{10}=x^6$

$y^5=x^3$

taking $z\to 0$ along x-axis $(y=0)$ then.

$$f'(0)=\lim_{x\to 0}\frac{x^3\times 0}{x^6+0}=0$$

Now take $z\to 0$ along to the curve $y^5=x^3$ then

$$f'(0)=\lim_{x\to 0}\frac{x^6}{x^6+x^6}=\frac{1}{2}$$

$\rightarrow f'(0)$ does not exist uniquely along any path $\Rightarrow f(z)$ is not analytic at origin but C-R equations are satisfied at origin.

Ques : Prove that the function defined by

$$f(z)=\begin{cases}\dfrac{x^2(1+i)-y^3(1-i)}{x^2+y^2}, & z\neq 0\\ 0, & z=0\end{cases}$$

Sol^n : Proceed as Above.

Ques : Give an example of a function in which C-R equations are satisfied yet the function is not analytic at the origin. Justify your answer.

Sol^n : Same as Above

Ques : Show that the function defined by $\sqrt{|xy|}$ is not analytic at origin, although C-R eq's are satisfied (2015-16) (2016-17)

Sol^n : let $f(z) = u+iv = \sqrt{|xy|}$

here $u = \sqrt{|xy|}$, $v=0$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,k)-v(0,0)}{k} = 0$$

it is clear that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Now $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{z}$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|xy|}}{(x+iy)}$$

let $z\to 0$ along to the line $y=mx$, then

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}-0}{x(1+im)} = \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

Thus the limit on RHS depends upon m & hence will different values for different values on m, Therefore, f'(0) is not unique.

Hence the function f(z) is not analytic at z=0.

Ques : Using C-R equations, show that $f(z) = |z|^2$ is not analytic at any point. (2012-13)

Sol^n : For practice.

Ques : If $f(z)$ is an analytic function of z, then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 = 2|f'(z)|^2$ (2011-12)

Sol^n : $f(z) = u + iv \Rightarrow Re\, f(z) = u =$ Real part of $f(z)$

$$|Re\, f(z)|^2 = |u|^2 = u^2$$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 0$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2\right] \quad \text{using CR eq}^n$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 2|f'(z)|^2$$

$$= RHS.$$

Cauchy-Riemann equations in Polar form

we know that $f(z) = f(x+iy) = f[r(\cos\theta + i\sin\theta)]$
$= f[re^{i\theta}] = u+iv$

then CR equations in Polar form is

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \& \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \checkmark$$

where $x = r\cos\theta$, $y = r\sin\theta$

Derivative of $w = f(z)$ in Polar form:

$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$

OR $$\frac{dw}{\partial z} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}$$

Ques: Write the Cauchy's Reimann equations in Polar coordinates. (2015-16), (2017-18).

Sol^n: See Above.

Ques: Find p given that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ is analytic.

Sol^n: $f(z) = u+iv = r^2\cos 2\theta + ir^2\sin p\theta$

$\Rightarrow u = r^2\cos 2\theta$, $v = r^2\sin p\theta$

$\frac{\partial u}{\partial r} = 2r\cos 2\theta$, $\frac{\partial v}{\partial r} = 2r\sin p\theta$, $\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta$

$\frac{\partial v}{\partial \theta} = pr^2\cos p\theta$.

given that $f(z)$ is analytic

so $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ [C-R eqⁿs]

$2r\cos 2\theta = pr\cos p\theta$. & $r r \sin p\theta = 2r\sin 2\theta$

Both these equations are satisfied if $p=2$.

Harmonic function ⇒ A function of x & y which satisfies Laplace equation is known as Harmonic function.

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is known as Laplacian operator.

& $\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as Laplace equation.

Result ⇒ If $f(z) = u + iv$ is an analytic function then u & v both are harmonic functions.

Note ⇒ If $u + iv = f(z)$ is an analytic functions, then u & v are known as Harmonic conjugate of each other.

Ques : Define Harmonic function.

Sol^n : See Above.

Ques : Show that $v(x,y) = e^{-x}(x\cos y + y\sin y)$ is Harmonic. find its Harmonic Conjugate.

(2013-14)

Sol^n : let $v = e^{-x}(x\cos y + y\sin y)$

if $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ then v will be Harmonic

$\frac{\partial v}{\partial x} = e^{-x}[\cos y + 0] + (x\cos y + y\sin y)(-e^{-x})$

$= e^{-x}[\cos y - x\cos y - y\sin y]$

$$\frac{\partial^2 v}{\partial x^2} = e^{-x}[-\cos y] + [\cos y - x\cos y - y\sin y](-e^{-x})$$

$$= e^{-x}[-\cos y - \cos y + x\cos y + y\sin y]$$

$$= e^{-x}[-2\cos y + x\cos y + y\sin y] \quad \leftarrow (1)$$

$$\frac{\partial v}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^{-x}[-x\cos y + y \times -\sin y + \cos y + \cos y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^{-x}[-x\cos y - y\sin y + 2\cos y] \quad - (2)$$

Adding (1) & (2)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^{-x}[-2\cos y + x\cos y + y\sin y - x\cos y - y\sin y + 2\cos y]$$

$$= 0$$

$\Rightarrow$ $v(x,y)$ is a Harmonic function.

Harmonic conjugate of $v(x,y)$ $\Rightarrow$ After next topic.

Milne-Thomson Method

By this method we can find $f(z)$ when only u or v is given.

we have following cases:-

Case-I → When only $u(x,y)$ is given:

$$f(z) = \int \left[\phi_1(z,0) - i\,\phi_2(z,0) \right] dz + c$$

where $\phi_1(x,y) = \frac{\partial u}{\partial x}$ & $\phi_2(x,y) = \frac{\partial u}{\partial y}$

$\phi_1(z,0) = \left[\phi_1(x,y)\right]_{\substack{at\ x=z \\ \& y=0}}$ $\quad \phi_2(z,0) = \left[\phi_2(x,y)\right]_{\substack{at\ x=z \\ \& y=0}}$

Case-II → When only imaginary part $v(x,y)$ is given:

$$f(z) = \int \left[\psi_1(z,0) + i\,\psi_2(z,0) \right] dz + c$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ & $\psi_2(x,y) = \frac{\partial v}{\partial x}$

$\psi_1(z,0) = \left[\psi_1(x,y)\right]_{\substack{at\ x=z \\ \& y=0}}$ & $\psi_2(z,0) = \left[\psi_2(x,y)\right]_{\substack{at\ x=z \\ \& y=0}}$

Case-III → When $u \cdot v$ is given ⇒ let $f(z) = u + iv$ — ①

$if(z) = iu - v$ — ②

Adding ① & ②

$(1+i)f(z) = (u-v) + i(u+v)$ — ③

$F(z) = U + iV$

where $U = u-v$, $V = u+v$, $F(z) = (1+i)f(z)$

Now if $U = u-v$ (Real part) is given.

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c \quad \text{— ④}$$

where $\phi_1(z,0) = \left[\phi_1(x,y) = \frac{\partial U}{\partial x}\right]_{\text{at } x=z \,\&\, y=0}$

$\phi_2(z,0) = \left[\phi_2(x,y) = \frac{\partial U}{\partial y}\right]_{\text{at } x=z \,\&\, y=0}$

After solving ④

$f(z) = \frac{F(z)}{(1+i)}$ is Required analytic function.

Case-4 → When $u+v$ is given →

when $V = u+v$ (Imaginary part) is given.

then $F(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz + c$

where $\psi_1(z,0) = \left[\psi_1(x,y) = \frac{\partial V}{\partial y}\right]_{\text{at } x=z \,\&\, y=0}$

$\psi_2(z,0) = \left[\psi_2(x,y) = \frac{\partial V}{\partial x}\right]_{\text{at } x=z \,\&\, y=0}$

then $f(z) = \frac{F(z)}{(1+i)}$

Ques : Show that $v(x,y) = e^{-x}(x\cos y + y\sin y)$ is Harmonic. find its Harmonic conjugate. (2013-14)

Sol^n : Harmonic ⇒ Already prooved.

here $v = e^{-x}(x\cos y + y\sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y) = \phi_1(x,y)$$

$$\frac{\partial v}{\partial x} = e^{-x}\cos y - e^{-x}(x\cos y + y\sin y) = \phi_2(x,y)$$

$$\phi_1(z,0) = 0$$

$$\phi_2(z,0) = e^{-z} - e^{-z}z = (1-z)e^{-z}$$

By Milne Thomson Method:

$$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz + c$$

$$= i\int (1-z)e^{-z}\,dz + c$$

$$= i\left[(1-z)(-e^{-z}) - \int(-1)(-e^{-z})\,dz\right] + c$$

$$= i\left[(z-1)e^{-z} + e^{-z}\right] + c$$

$$= i\left[ze^{-z} \cancel{-e^{-z}} \cancel{+e^{-z}}\right] + c$$

$$= ize^{-z} + c \quad \text{—— (1)}$$

which is ~~required~~ analytic function.

$$f(z) = i(x+iy)e^{-(x+iy)} + c$$

$$= (ix-y)e^{-x}.e^{-iy} + c = (ix-y)e^{-x}(\cos y - i\sin y) + c$$

$$= ie^{-x}x\cos y + xe^{-x}\sin y - ye^{-x}\cos y + iye^{-x}\sin y + c$$

$$= e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y) + c$$

So Harmonic conjugate of $v(x,y)$ is

$u(x,y) = e^{x}(x\sin y - y\cos y)$

Ques : Define Harmonic function. Show that the function $v = \log(x^2+y^2) + x - 2y$ is Harmonic. Also find analytic function $f(z) = u+iv$. (2011-12)

Sol^n : <u>Harmonic function</u>: As Above

Now $v = \log(x^2+y^2) + x - 2y$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)\times 2 - 2x(2x)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$= \frac{2(y^2-x^2)}{(x^2+y^2)^2} \quad \text{--- } (1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - 2$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)\times 2 - 2y(2y)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$= -\frac{2(y^2-x^2)}{(x^2+y^2)^2} \quad \text{--- } (2)$$

Adding (1) & (2)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v(x,y) \text{ is Harmonic function.}$$

let required analytic function is

$f(z) = u + iv$

$\phi_1(x,y) = \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2 \Rightarrow \phi_1(z,0) = 0 - 2 = -2$

$\phi_2(x,y) = \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 \Rightarrow \phi_2(z,0) = \frac{2z}{z^2} + 1 = \frac{2}{z} + 1$

By Milne thomson Method →

$f(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\,dz + c$

$= \int \left[-2 + i\left(\frac{2}{z} + 1\right)\right] dz + c$

$= -2z + 2i\log z + iz + c$

$= (i-2)\,z + 2i\log z + c$ Ans.

Ques : Show that $u = x^4 - 6x^2y^2 + y^4$ is Harmonic function. find complex function $f(z)$ whose u is a real part. (2018-19)

Sol : Given $u(x,y) = x^4 - 6x^2y^2 + y^4$

$u_x = 4x^3 - 12xy^2 = \phi_1(x,y)$

$u_y = -12x^2y + 4y^3 = \phi_2(x,y)$

$u_{xx} = 12x^2 - 12y^2$ — ①

$u_{yy} = -12x^2 + 12y^2$ — ①

adding ① & ②

$u_{xx} + u_{yy} = -12x^2 + 12y^2 + 12x^2 - 12y^2 = 0$

$\Rightarrow u(x,y)$ is a Harmonic function.

Now

$\phi_1(z,0) = 4z^3 - 0 = 4z^3$

$\phi_2(z,0) = 0$

By Milne Thomson Method

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] dz + c$

$= \int 4z^3 dz + c$

$= \frac{4z^4}{4} + c$

$= z^4 + c$

Ques: If $u = 3x^2y - y^3$ find analytic function $f(z) = u + iv$

Ans: $v = 3xy^2 - x^3$, $f(z) = -iz^3$

Ques: Determine analytic function $f(z) = u + iv$ in terms of z whose real part is $e^{x}(x\sin y - y\cos y)$ (2013-14)

Ans: $f(z) = ize^{-z} + c$

Ques: verify that the function $u(x,y) = xy$ is Harmonic & find the Harmonic conjugate. Express $u+iv$ as an analytic function $f(z)$ where $u = x^2 - y^2 - y$ (2015-16)

Ans: Harmonic conjugate $v = \frac{y^2 - x^2}{2}$, $f(z) = z^2 + iz$

Ques : Determine an analytic function $f(z)$ in terms of z if $u+v = \frac{2\sin 2x}{e^{2y}} + e^{2y} - 2\cos 2x$

Sol^n : let $f(z) = u+iv$ —①

$\Rightarrow if(z) = iu - v$ —②

Adding ① & ②

$$\left.\begin{aligned}(1+i)f(z) &= (u-v) + i(u+v) \\ \Rightarrow \quad F(z) &= U + iV\end{aligned}\right\} ③$$

where $U = u-v$ & $V = u+v$

$$\Rightarrow \quad V = u+v = \frac{2\sin 2x}{e^{2y}} + e^{2y} - 2\cos 2x$$

$$\phi_1(x,y) = \frac{\partial V}{\partial y} = 2\sin 2x(-2)e^{-2y} + 2e^{2y}$$

$$\phi_1(z,0) = -4\sin 2z + 2$$

$$\phi_2(x,y) = \frac{\partial V}{\partial x} = 2e^{-2y}.2\cos 2x + 0 + 4\sin 2x$$

$$\phi_2(z,0) = 4\cos 2z + 4\sin 2z$$

By Milne-Thomson Method

$$F(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]dz + c$$

$$= \int [-4\sin 2z + 2 + i(4\cos 2z + 4\sin 2z)]dz$$

$$= \int [2 - 4(1-i)\sin 2z + 4i\cos 2z]dz + c$$

$$= \left[2z + 4(1-i)\frac{\cos 2z}{2} + 4i\frac{\sin 2z}{2}\right] + c$$

$$= 2[z + (1-i)\cos 2z + i\sin 2z] + c$$

$$F(z) = (1+i)f(z) = 2[z + (1-i)\cos 2z + i\sin 2z] + c$$

$$f(z) = \frac{2}{(1+i)}[z + (1-i)\cos 2z + i\sin 2z] + \frac{c}{(1+i)}$$

let $\frac{c}{1+i} = c_1$, $\frac{1}{(1+i)}\frac{(1-i)}{(1-i)} = \frac{1-i}{1+1} = \frac{1}{2}(1-i)$

so $f(z) = 2 \times \frac{1}{2}(1-i)[z + (1-i)\cos 2z + i\sin 2z] + c_1$

$$= (1-i)z + (1-1-2i)\cos 2z + (i+1)\sin 2z + c_1$$

$$= (1-i)z - 2i\cos 2z + (1+i)\sin 2z + c_1$$

Ques: In a 2-D fluid flow the stream function is $\psi = \frac{-y}{x^2+y^2}$, find the velocity potential function ϕ.

Sol^n: Ques. for practice, velocity potential = Harmonic conjugate $= \frac{x}{x^2+y^2}$ Ans.

Ques : Determine Analytic function $f(z)=u+iv$ in terms of z where $u-v = e^x(\cos y - \sin y)$ (2015-16)

Sol^n : As above

$$F(z) = U + iV$$

where $F(z) = (1+i)f(z)$, $U = u-v$, $V = u+v$

Here $U = u-v = e^x(\cos y - \sin y)$ = Real part

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = (\cos y - \sin y)e^x$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$\phi_1(z,0) = (\cos 0 - \sin 0)\cdot e^z = e^z$$

$$\phi_2(z,0) = e^z(-\sin 0 - \cos 0) = -e^z$$

By Milne Thomson Method

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)dz]dz + c$$

$$= \int [e^z + ie^z]dz + c = (1+i)\int e^z dz + c$$

$$= (1+i)e^z + c$$

$$F(z) = (1+i)f(z) = (1+i)e^z + c$$

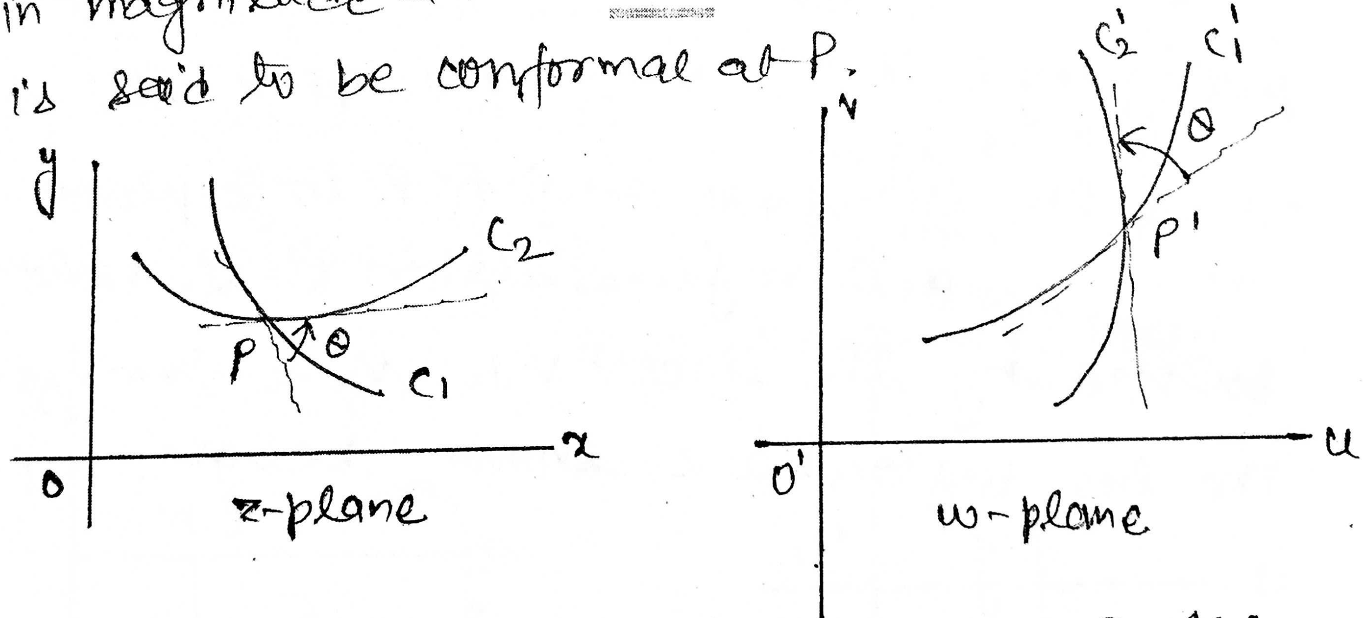
$$f(z) = e^z + \frac{c}{1+i}$$

$$f(z) = e^z + c_1 \quad \text{Ans.}$$

Conformal Mapping

Mapping of z-plane into w-plane ⇒ for every point (x,y) in the z-plane, the relation $w=f(z)$ defines a corresponding point (u,v) in the w-plane then $w=f(z)$ is known as the mapping or transformation from z-plane into w-plane.

Conformal Mapping ⇒ Let two curves C_1, C_2 in the z-plane intersects at the point P & the corresponding curves C_1', C_2' in the w-plane intersects at P' under the transformation $w=f(z)$. If the angle of intersection of the curves at P is same as the angle of intersection of the curves at P', both in magnitude & sense, then the transformation is said to be conformal at P.



Definition ⇒ A transformation which preserves angles both in magnitude & sense between every pair of curves through a point is said to be conformal at the point.

(2018-19)

(2) Rotation: $w = ze^{i\theta_0}$,

if $\theta_0 > 0$, Rotation is anticlockwise

if $\theta_0 < 0$, Rotation is clockwise

Ques: Consider the transformation $w = ze^{i\pi/4}$ & determine the region R' in w-plane corresponding to the triangular region R bounded by the lines $x=0$, $y=0$ & $x+y=1$ in z-plane.

Sol^n: $w = ze^{i\pi/4} \Rightarrow u+iv = (x+iy)\left(\frac{1+i}{\sqrt{2}}\right)$

$\Rightarrow u = \frac{1}{\sqrt{2}}(x-y)$, $v = \frac{1}{\sqrt{2}}(x+y)$

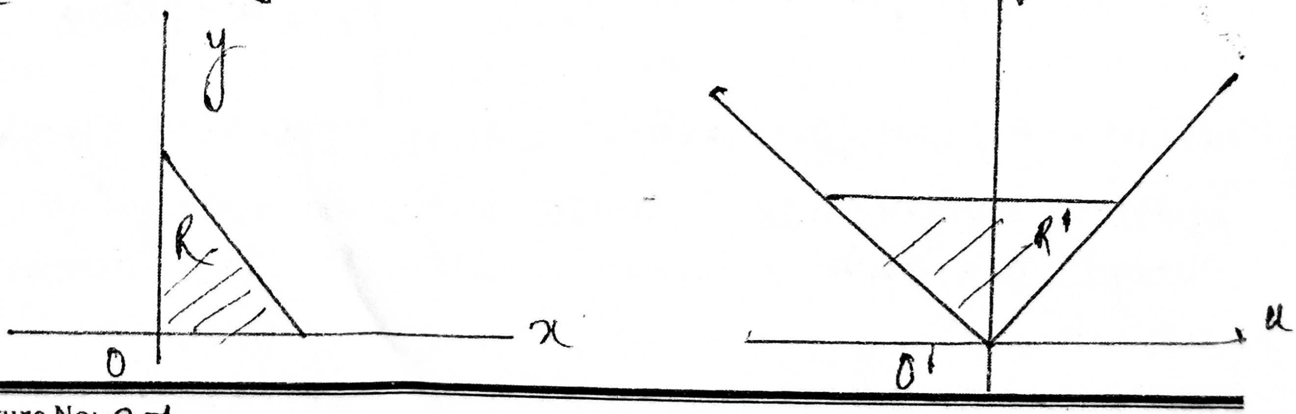
put $x=0 \Rightarrow u = -\frac{1}{\sqrt{2}}y$, $v = \frac{1}{\sqrt{2}}y$ or $\boxed{u=-v}$

put $y=0 \Rightarrow u = \frac{1}{\sqrt{2}}x$, $v = \frac{1}{\sqrt{2}}x$ or $\boxed{u=v}$

putting $x+y=1$, in (1) , $v = \frac{1}{\sqrt{2}}$

Hence the triangular region R in z-plane is mapped on a triangular region R' of w plane bounded by the lines $v=u$, $v=-u$, $v=\frac{1}{\sqrt{2}}$.

The two regions are shown below:



Some Standard Transformation:

① Translation: $w = z + c$, where c is a complex variable

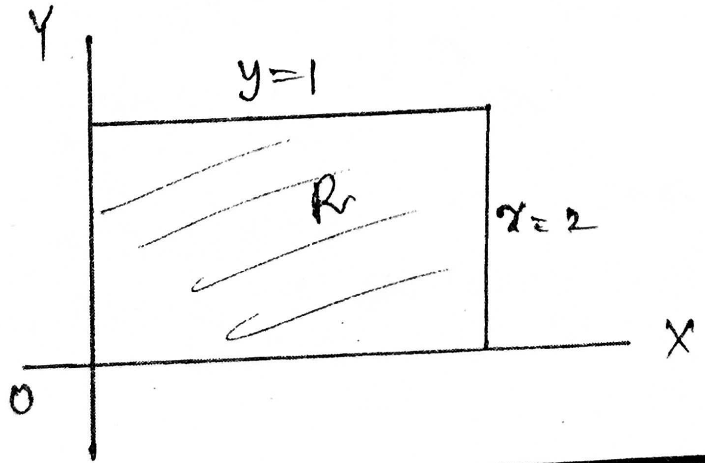
Ques: Let a rectangular domain R be bounded by $x=0, y=0, x=2, y=1$. Determine the region R' of w-plane into which R is mapped under the transformation $w = z + (1-2i)$

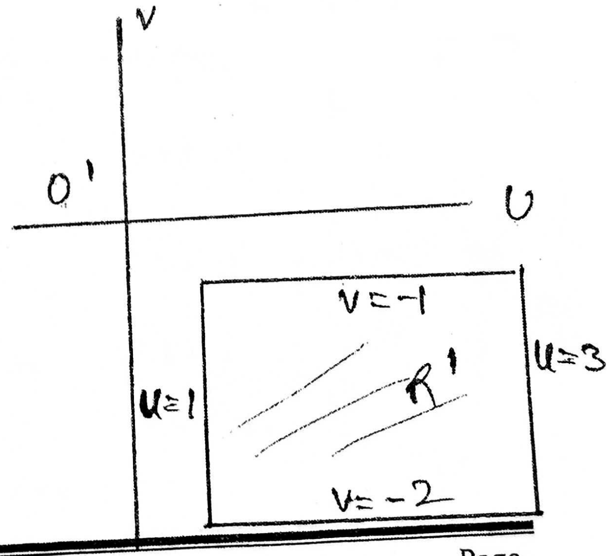
Soln:
$$w = z + (1-2i)$$
$$u + iv = x + iy + 1 - 2i = (x+1) + i(y-2)$$
$$\Rightarrow u = x+1 \ \& \ v = y-2$$

By the map $u = x+1$, the lines $x=0$, $x=2$ are mapped respectively on the lines $u=1$, $u=3$.

By the map $v = y-2$, the lines $y=0$, $y=1$ are mapped on $v=-2$ & $v=-1$ respectively.

Then required image is rectangle R' in w-plane bounded by $u=1$, $u=3$, $v=-2$ & $v=-1$





(3) Magnification (stretching)

$w = az$ where a is real.

Ques: consider the transformation $w = 2z$ & determine the region R' of w-plane into which the triangular region R enclosed by the lines $x=0$, $y=0$ & $x+y=1$ in the z-plane is mapped under the map.

Soln: $w = 2z \Rightarrow u+iv = 2(x+iy)$

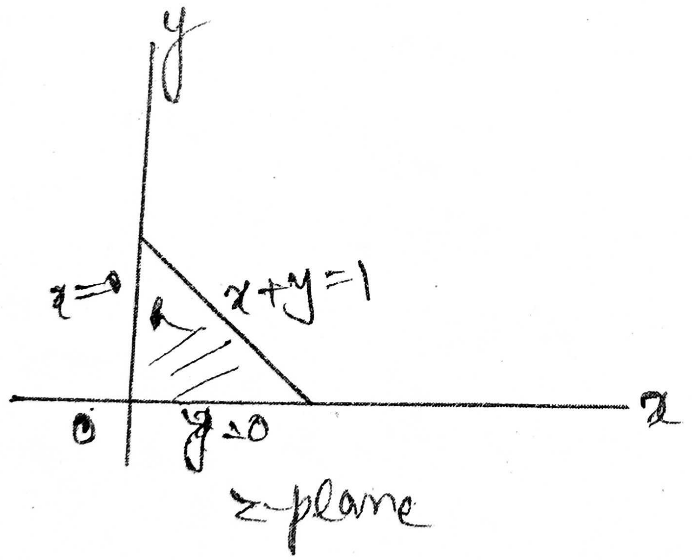
$\Rightarrow u = 2x$ & $v = 2y$

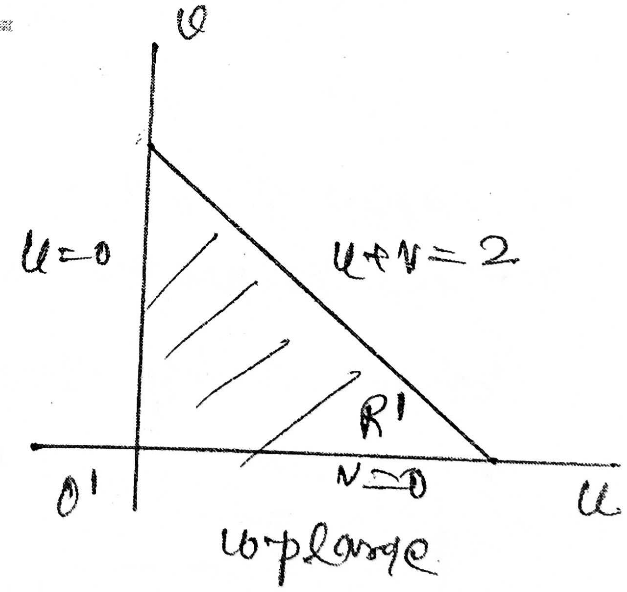
$x=0 \Rightarrow u=0$, $y=0 \Rightarrow v=0$

$x+y=1 \Rightarrow u=2x,\ v=2y \Rightarrow u+v=2.$

Hence R' is triangular region bounded by

$u=0,\ v=0,\ u+v=2$





Bilinear Transformation

A transformation of the form

$w = \frac{az+b}{cz+d}$, where a, b, c, d are complex constants

& $ad - bc \neq 0$, is called Bilinear or Mobius Transformation.

Invariant or Fixed Points of Bilinear transformation

The points which coincide with their transformation are called invariant or fixed points of the transformation.

fixed points of $w = f(z)$ are obtain by putting $w = z$

i.e $z = \frac{az+b}{cz+d}$, gives fixed points of $w = \frac{az+b}{cz+d}$

Ques: find the fixed points of the bilinear transformation $w = \frac{2z-5}{z+4}$

Sol^n: $w = \frac{2z-5}{z+4}$ (given),

to find fixed points, let $w = z$

$\Rightarrow z = \frac{2z-5}{z+4} \Rightarrow z^2 + 2z + 5 = 0$

$\Rightarrow z = -1 \pm 2i$

$\Rightarrow z = -1+2i$ & $-1-2i$ are fixed points of bilinear transformation $w = \frac{2z-5}{z+4}$

Cross-Ratio ⇒ If there are four points z_1, z_2, z_3, z_4 taken in order then the ratio

$$\frac{(z_1-z_2)}{(z_2-z_3)} \cdot \frac{(z_3-z_4)}{(z_4-z_1)}$$ is called cross ratio of z_1, z_2, z_3 & z_4.

Properties of Bilinear Transformation:

① A bilinear transformation maps circles into circles.

② A bilinear transformation preserves cross ratio of four points.

i.e If z_1, z_2, z_3, z_4 maps on to w_1, w_2, w_3 & w_4 of w-plane respectively then

$$\frac{(w_1-w_2)}{(w_2-w_3)} \cdot \frac{(w_3-w_4)}{(w_4-w_1)} = \frac{(z_1-z_2)}{(z_2-z_3)} \cdot \frac{(z_3-z_4)}{(z_4-z_1)}$$

[which is also a method to find a bilinear transformation]

Ques: find the bilinear transformation which maps the points $z=1, i, -1$ onto the points $w=i, 0, -i$. Hence find the image of $|z|<1$.

Sol: Let $z_1 = z$ & $w_1 = w$

$z_2 = 1$ $\quad$ $w_2 = i$

$z_3 = i$ $\quad$ $w_2 = 0$

$z_4 = -1$ $\quad$ $w_4 = -i$

now applying cross-ratio

$$\frac{(z-z_1)}{(z_1-z_2)}\cdot\frac{(z_2-z_3)}{(z_3-z)} = \frac{(w-w_1)}{(w_1-w_2)}\cdot\frac{(w_2-w_3)}{(w_3-w)}$$

$$\frac{(z-1)}{(z+1)}\cdot\frac{(i+1)}{(i-1)} = \frac{(w-i)(i)}{(w+i)(-i)}$$

$$\frac{w-i+w+i}{w+i-w-i}$$

$$\frac{w-i}{w+i} = i\left(\frac{z-1}{z+1}\right)$$

$$\frac{2w}{-2i} = \frac{iz-i+z+1}{iz-i-z-1} = \frac{(1+i)z-(i-1)}{(i-1)z-(i+1)} = -i\left(\frac{z-i}{z+i}\right)$$

(Applying C & D Rule)

$$\Rightarrow \quad w = \frac{i-z}{i+z} \qquad \text{--- (1)}$$

(1) is the required bilinear transformation.

(1) can be written as $z = i\left(\frac{1-w}{1+w}\right)$

now $|z|<1$ is mapped into the region.

$$\left|i\left(\frac{1-w}{1+w}\right)\right| < 1 \quad \Rightarrow \quad \frac{|i||1-w|}{|1+w|} < 1$$

$$|1-w| < |1+w| \quad \Rightarrow \quad |1-u-iv| < |1+u+iv|$$

$$(1-u)^2+v^2 < (1+u)^2+v^2$$

$$1+u^2-2u+v^2 < 1+u^2+2u+v^2$$

$$-2u < 2u$$

$$\Rightarrow \quad u > 0$$

Hence $|z|<1$ mapped into the entire half of the w-plane to the right of the Imaginary axis.

Ques: Find the bilinear transformation which maps the points $z = 0, -i, 2i$ onto the points $w = 5i, \infty, -\frac{i}{3}$ respectively. (2018-19)

Sol^n Given $z_1 = z$ & $w_1 = w$

$z_2 = 0$ $\quad w_2 = 5i$

$z_3 = -i$ $\quad w_3 = \infty$

$z_4 = 2i$ $\quad w_4 = -\frac{i}{3}$

By using the cross ratio

$$\frac{w_1-w_2}{w_2-w_3}\cdot\frac{w_3-w_4}{w_4-w_1} = \frac{z_1-z_2}{z_2-z_3}\cdot\frac{z_3-z_4}{z_4-z_1}$$

$$\frac{w_1-w_2}{\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}\cdot\frac{\cancel{w_3}\left(1-\frac{w_4}{w_3}\right)}{(w_4-w_1)} = \frac{z_1-z_2}{z_2-z_3}\cdot\frac{z_3-z_4}{z_4-z_1}$$

$$\frac{w-5i}{\left(\frac{5i}{\infty}-1\right)}\cdot\frac{\left(1-\frac{(-i/3)}{\infty}\right)}{\left(-\frac{i}{3}-w\right)} = \left(\frac{z-0}{0+i}\right)\left(\frac{-i-2i}{2i-z}\right)$$

$$\frac{w-5i}{w+\frac{i}{3}} = \frac{z(-3i)}{(z-2i)(-i)}$$

$$\frac{\cancel{3}(w-5i)}{3w+i} = \frac{\cancel{3}z}{z-2i}$$

$$\frac{w-5i}{3w+i} = \frac{z}{z-2i} \Rightarrow (w-5i)(z-2i) = z(3w+i)$$

$$wz-5iz-2wi+10i^2 = 3zw+zi$$

$$w(-2z-2i) = zi(1+5)+10$$

$$w = \frac{6iz+10}{-2(z+i)} \Rightarrow -\frac{(3z-5i)}{1-iz} = \frac{3z-5i}{iz-1}$$

Ans.